APPLICATION OF SOMMERFELD—MALYUZHINETS INTEGRAL TO DIFFUSION PROBLEMS IN WEDGE-SHAPED REGIONS WITH INHOMOGENEOUS BOUNDARY CONDITIONS OF THE FIRST AND SECOND KIND

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A Sommerfeld —Malyuzhinets integral representation is found which solves unsteady diffusion problems in wedge-shaped regions.

In the present paper we investigate unsteady diffusion problems (parabolic or hyperbolic equations) for zero-value initial conditions in wedge-shaped regions in the case of boundary conditions of the first and second kind in the presence or absence of a first-order chemical reaction. After applying the Laplace transform [1], these problems can be written in the form

$$L[v(r,\ \varphi;\ \mu)]=0,\quad 0\leqslant r\leqslant\infty,\quad -\Phi\leqslant\varphi\leqslant\Phi, \tag{1}$$

$$v<\infty,\quad r=0,\quad -\Phi\leqslant\varphi\leqslant\Phi, \tag{2}$$

$$v=0,\quad r=\infty,\quad -\Phi\leqslant\varphi\leqslant\Phi, \tag{3}$$

$$v=F_1^{\mp}(r,\ \mu)\quad \text{or}\quad \frac{1}{\mu r}\frac{\partial v}{\partial\varphi}=F_2^{\mp}(r,\ \mu);\quad 0\leqslant r\leqslant\infty,\quad \varphi=\mp\Phi, \tag{4}$$

where $L = (1/r)(\partial/\partial r)\,[r(\partial/\partial r)] + (1/r^2)(\partial^2/\partial\varphi^2) - \mu^2$; μ is a complex number. The solution of problem (1)-(4) can be sought in the form of a Sommerfeld —Malyuzhinets integral [2-11], i.e., in the form

$$v(r,\ \varphi;\ \mu)=\frac{1}{2\pi i}\int\limits_{\gamma}\exp\{\mu r\cos(\varphi-\alpha)\}\,H(\alpha)\,d\alpha. \tag{5}$$

Here the kernel $\exp\{\mu r\cos(\varphi-\alpha)\}$ satisfies differential equation (1); the contour γ must be such that boundary conditions (2)-(3) are satisfied; the function $H(\alpha,\ \Phi)$ is such that condition (4) is satisfied.

In order to fix the contour γ it is sufficient to make the following change of variable:

$$-z=\varphi-\alpha;\quad dz=d\alpha. \tag{6}$$

Using (6) we bring (5) to the form

$$v(r,\ \varphi;\ \mu)=\frac{1}{2\pi i}\int\limits_{\gamma}\exp\{\mu r\cos z\}\,H(z+\varphi)\,dz. \tag{7}$$

The inhomogeneous parts of boundary conditions (4) — the functions $F_1^{\mp}(r,\mu)$ and $F_2^{\mp}(r,\mu)$ — can be expressed with the aid of the Malyuzhinets transform [12]:

$$F_j^{\mp}(r,\ \mu)=\frac{1}{2\pi i}\int\limits_{\gamma}\exp\{\mu r\cos z\}\,f_j^{\mp}(z,\ \mu)\,dz;\quad j=1,\ 2; \tag{8}$$

$$f_j^{\mp}(z,\ \mu)=-\frac{\mu\sin z}{2}\int\limits_{0}^{\infty}\exp\{-\mu r\cos z\}\,F_j^{\mp}(r,\ \mu)\,dr;\quad j=1,\ 2. \tag{9}$$

Inserting expressions (7) and (8) into boundary conditions (4) and remembering that

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$$\frac{\partial v(r,\ \varphi;\ \mu)}{\partial\varphi}=\frac{\mu r}{2\pi i}\int\limits_{\gamma}\sin z\exp\{\mu r\cos z\}\,H(z+\varphi)\,dz, \tag{10}$$

we obtain the following for boundary conditions of the first kind for $\varphi = \mp\Phi$:

$$\frac{1}{2\pi i}\int\limits_{\gamma}\exp\{\mu r\cos z\}\,[H(z+\varphi)-f_1^{\mp}(z,\ \mu)]\,dz=0, \tag{11}$$

and the following for boundary conditions of the second kind:

$$\frac{1}{2\pi i}\int\limits_{\gamma}\exp\{\mu r\cos z\}\,[\sin zH(z+\varphi)-f_2^{\mp}(z,\ \mu)]\,dz=0. \tag{12}$$

A necessary and sufficient condition for integrals (11) and (12) to equal zero is that the expressions in the square brackets be even [8,9]; from this we find the following functional equations corresponding to equations (11) and (12):

$$H(z\mp\Phi)-H(-z\mp\Phi)=2f_1^{\mp}(z,\ \mu), \tag{13}$$

$$H(z\mp\Phi)+H(-z\mp\Phi)=2f_2^{\mp}(z,\ \mu)/\sin z. \tag{14}$$

For mixed boundary conditions, for instance, the first kind for $\varphi = +\Phi$ and the second kind for $\varphi = -\Phi$, we obtain the following equations:

$$\begin{aligned}H(z+\Phi)-H(-z+\Phi)&=2f_1^{+}(z,\ \mu),\\ H(z-\Phi)+H(-z-\Phi)&=2f_2^{-}(z,\ \mu).\end{aligned} \tag{15}$$

All three systems of equations (13)-(15) can be written compactly in the form

$$\begin{aligned}H(z+\Phi)-\varepsilon_1H(-z+\Phi)&=Q^{+}(z),\\ H(z-\Phi)-\varepsilon_2H(-z-\Phi)&=Q^{-}(z).\end{aligned} \tag{16}$$

Here for $v = F_1^{\pm}(r, \mu)$ and $\varphi = \pm\Phi$, the quantities $\varepsilon_1 = \varepsilon_2 = 1$ and

$$Q^{+}(z)=2f_1^{+}(z,\ \mu),\quad Q^{-}(z)=2f_1^{-}(z,\ \mu);$$

for $(1/\mu r)(\partial v/\partial\varphi) = F_2^{\pm}(r, \mu)$ and $\varphi = \pm\Phi$, the quantities $\varepsilon_1 = \varepsilon_2 = -1$ and

$$Q^{+}(z)=2f_2^{+}(z,\ \mu)/\sin z,\quad Q^{-}(z)=2f_2^{-}(z,\ \mu)/\sin z;$$

for $v = F_1^{+}(r, \mu)$ and $\varphi = +\Phi$, and for $(1/\mu r)(\partial v/\partial r) = F_2^{-}(r, \mu)$ and $\varphi = -\Phi$, the quantities $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and

$$Q^{+}(z)=2f_1^{+}(z,\ \mu),\quad Q^{-}(z)=2f_2^{-}(z,\ \mu)/\sin z.$$

Following [13-18] we seek the solution of functional equations (16) in the form

$$H(z)=u(z)\,\sigma(z). \tag{17}$$

Inserting (17) into (16) gives the following functional equations

$$\sigma(z\pm\Phi)-\sigma(z\pm\Phi)=Q^{+}(z)/u(z\pm\Phi), \tag{18}$$

$$\begin{aligned}u(z+\Phi)-\varepsilon_1u(-z+\Phi)&=0,\\ u(z-\Phi)-\varepsilon_2u(-z-\Phi)&=0.\end{aligned} \tag{19}$$

We have for the solution of Eqs. (19) [18]

$$\begin{aligned}&\varepsilon_1=\varepsilon_2=1,\quad u(z)=1,\quad \varepsilon_1=\varepsilon_2=-1,\quad u(z)=\cos(\pi z/2\Phi),\\ &\varepsilon_1=1,\quad \varepsilon_2=-1,\quad u(z)=\sin[\pi(z+\Phi)/4\Phi],\end{aligned} \tag{20}$$

and the solution of (18) we seek in the form of a sum, i.e.,

$$\sigma(z)=\sigma_1(z)+\sigma_2(z-2\Phi). \tag{21}$$

Inserting expression (21) into (18) leads to the following inhomogeneous functional equations:

$$\sigma_1(z+\Phi)-\sigma_1(-z+\Phi)=Q^+(z)/u(z+\Phi),$$
$$\sigma_1(z-\Phi)-\sigma_1(-z-\Phi)=0, \tag{22}$$
$$\sigma_2(z+\Phi)-\sigma_2(-z+\Phi)=Q^-(z)/u(z-\Phi),$$
$$\sigma_2(z-\Phi)-\sigma_2(-z-\Phi)=0. \tag{23}$$

We note that $Q^+(z)/u(z+\Phi)$ and $Q^-(z)/u(z-\Phi)$ are always odd functions. As shown by Tuzhilin [18], the solutions of functional equations (22) and (23) can be written, respectively, in the form

$$\sigma_1(z)=\sin^n\left(\frac{\pi z}{2\Phi}\right)\frac{i}{8\Phi}\int\limits_{-i\infty}^{i\infty}\frac{Q^+(\tau)\sin(\pi\tau/2\Phi)\,d\tau}{u(\tau+\Phi)\cos^n\left(\frac{\pi\tau}{2\Phi}\right)\left[\cos\left(\frac{\pi\tau}{2\Phi}\right)-\sin\left(\frac{\pi z}{2\Phi}\right)\right]}, \tag{24}$$

$$\sigma_2(z)=\sin^m\left(\frac{\pi z}{2\Phi}\right)\frac{i}{8\Phi}\int\limits_{-i\infty}^{i\infty}\frac{Q^-(\tau)\sin(\pi\tau/2\Phi)\,d\tau}{u(\tau-\Phi)\cos^m\left(\frac{\pi\tau}{2\Phi}\right)\left[\cos\left(\frac{\pi\tau}{2\Phi}\right)-\sin\left(\frac{\pi z}{2\Phi}\right)\right]}, \tag{25}$$

where n and m are numbers such that $Q^+(\tau)/[u(\tau+\Phi)\cos^n(\pi\tau/2\Phi)]$ and $Q^-(\tau)/[u(\tau-\Phi)\cos^m(\pi\tau/2\Phi)]$ decrease exponentially for $|I_m(\tau)|\to\infty$ and $\mathrm{Re}(\tau)=0$.

Utilizing expressions (17) and (21), solution (7) can be expressed as follows:

$$v(r,\varphi;\mu)=\frac{1}{2\pi i}\int\limits_{\gamma}\exp\{\mu r\cos z\}\,u(z+\varphi)[\sigma_1(z+\varphi)+\sigma_2(z+\varphi-2\Phi)]\,dz. \tag{26}$$

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